

Poisson + Gamma => NB

Q&A

1. Definitions and Distributions
2. Mix Distributions
3. Examples

Poisson Distribution:

$$N \sim \text{Pois}(\lambda) \quad \text{then} \quad P(N = k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad k = 0, 1, 2, \dots$$

N : number of customers, claims or patients.

Exponential Distribution:

$$X \sim \text{Exp}(\theta) \quad \text{then} \quad f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x > 0 \\ 0 & o/w \end{cases}$$

X : Claim size or interarrival times.

Gamma Distribution:

$$X \sim \text{Gamma}(\alpha, \theta) \quad \text{then} \quad f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-\frac{x}{\theta}}}{\theta^{\alpha} \Gamma(\alpha)} & x > 0 \\ 0 & o/w \end{cases}$$

X : Claim size or sum of exponentially distributed random variables.

Mix Distributions

$$\text{Pois}(\lambda) + \text{Exp}(\theta) = \text{Geo}(p) \quad p = \frac{1}{1+\theta}$$

$$\text{Pois}(\lambda) + \text{Gamma}(\alpha, \theta) = \text{NB}(r = \alpha, \beta = \theta) \quad p = \frac{1}{1+\beta}$$

Q1. Let number of claims, N , be a Poisson distributed random variable with parameter λ , and Λ is also a random variable distributed exponentially with parameter θ .

- i. $E[N]$
- ii. $\text{Var}[N]$
- iii. $P(N=0)$
- iv. $P(N=2)$

- i. $E[N] = E[E[N | \Lambda]] = E[\Lambda] = \theta$
- ii. $\text{Var}[N] = E[V[N | \Lambda]] + V[E[N | \Lambda]] = E[\Lambda] + V[\Lambda] = \theta + \theta^2$

iii. $P(N=0) = \int_0^{\infty} e^{-\lambda} \frac{1}{\theta} e^{-\frac{\lambda}{\theta}} d\lambda = \frac{1}{1+\theta}$

$$N \sim \text{Geo}(\theta) \rightarrow P(N=0) = p = \frac{1}{1+\theta}$$

iv. $P(N=2) = q^2 p = \left(\frac{\theta}{1+\theta}\right)^2 \frac{1}{1+\theta}$

Q2. Let number of claims, N , be a Poisson distributed random variable with parameter λ , and Λ is also a random variable distributed *Gamma* (α, θ).

- i. $E[N]$
- ii. $\text{Var}[N]$
- iii. $P(N=0)$
- iv. $P(N=2)$

i. $E[N] = E[E[N | \Lambda]] = E[\Lambda] = \alpha\theta$

ii. $\text{Var}[N] = E[V[N | \Lambda]] + V[E[N | \Lambda]] = E[\Lambda] + V[\Lambda] = \alpha\theta + \alpha\theta^2$

iii. $P(N=0) = \int_0^{\infty} e^{-\lambda} \frac{\lambda^{\alpha-1} e^{-\frac{\lambda}{\theta}}}{\theta^{\alpha} \Gamma(\alpha)} d\lambda = \binom{r+k-1}{k} \left(\frac{\theta}{1+\theta}\right)^k \left(\frac{1}{1+\theta}\right)^r = \left(\frac{1}{1+\theta}\right)^r$

$$N \sim \text{NB}(r = \alpha, \beta = \theta) \rightarrow P(N=0) = p^r = \left(\frac{1}{1+\theta}\right)^r$$

iv. $P(N=2) = \binom{r+2-1}{2} \left(\frac{\theta}{1+\theta}\right)^2 \left(\frac{1}{1+\theta}\right)^r$

Q3. Let number of claims, N , be a Poisson distributed random variable with parameter λ and prior distribution be a Gamma distribution with parameters $\alpha = 1$ and $\theta = 1.2$. There are 3 claims observed in the first year and no claim is observed in the second year. Find the Bühlmann Credibility premium for the following year.

$$\mu = E[N] = E[E[N|\Lambda]] = E[\Lambda] = 1 \times 1.2 = 1.2$$

$$v = E[V[N|\Lambda]] = E[\Lambda] = 1 \times 1.2 = 1.2$$

$$a = V[E[N|\Lambda]] = V[\Lambda] = 1 \times 1.2^2 = 1.44$$

$$Z = \frac{n}{n+k} \quad k = \frac{v}{a} = \frac{1.2}{1.44}$$

$$Z = \frac{n}{n+k} = \frac{2}{2+0.833} = 0.7059$$

$$C_p = Z\bar{X} + (1-Z)\mu = 0.7059(1.5) + (1-0.7059)(1.2) = 1.41.$$